

ℓ -adic Realization of Some Aspects of Landau-Ginzburg B -models*

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The Landau-Ginzburg B -model for a germ of a holomorphic function with an isolated critical point is constructed by K. Saito [10] and finished by M. Saito [11]. Douai and Sabbah construct the Landau-Ginzburg B -models for some Laurent polynomials [2, 3, 4]. The construction relies on analytic procedures, and one can not expect it can be done by purely algebraic method. In this note, we work out the ℓ -adic realization of the algebraic part of the construction. In §1, we define Frobenius type structures. One can consult [6, 7, 8] for details. In §2, we sketch the construction of the Landau-Ginzburg B -models for Laurent polynomials. Details can be found in [2, 3, 4, 6]. In §3, we study the ℓ -adic counterpart of the construction in §2.

1 Frobenius type structures

Let D be the germ of \mathbb{C} at 0, let X be a germ of complex manifold, and let \mathcal{F} be a trivial holomorphic vector bundle on $D \times X$. Denote the $\mathcal{O}_{D \times X}$ -module of holomorphic sections of \mathcal{F} also by \mathcal{F} . Let

$$\nabla : \mathcal{F}|_{(D-\{0\}) \times X} \rightarrow \mathcal{F}|_{(D-\{0\}) \times X} \otimes \Omega^1_{(D-\{0\}) \times X}$$

be an integrable connection. We say ∇ has a pole of *Poincaré rank* $\leq m$ along the divisor $X \times 0$ if

$$\nabla(\mathcal{F}) \subset \mathcal{F} \otimes \frac{1}{t^m} \left(\sum_i \mathcal{O}_{D \times X} dx_i + \mathcal{O}_{D \times X} \frac{dt}{t} \right),$$

where t is the coordinate for D , and (x_i) the coordinate for X .

Poincaré rank 0 case:

If the Poincaré rank is 0, we say (\mathcal{F}, ∇) has a *logarithmic pole* along $0 \times X$. Fix a global basis for \mathcal{F} , and write the connection matrix as

$$A = \sum_i \Omega_i(t, x) dx_i + \Omega(t, x) \frac{dt}{t}$$

for some matrices of holomorphic functions $\Omega_i(t, x)$ and $\Omega(t, x)$. We have

$$dA + A \wedge A = \sum_i \left(\frac{1}{t} \left(-\frac{\partial \Omega}{\partial x_i} + [\Omega, \Omega_i] \right) + \frac{\partial \Omega_i}{\partial t} \right) dt dx_i + \sum_{i < j} \left(\frac{\partial \Omega_j}{\partial x_i} - \frac{\partial \Omega_i}{\partial x_j} + [\Omega_i, \Omega_j] \right) dx_i dx_j.$$

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Since $dA + A \wedge A = 0$, we have

$$\begin{aligned} \frac{1}{t} \left(-\frac{\partial \Omega}{\partial x_i} + [\Omega, \Omega_i] \right) + \frac{\partial \Omega_i}{\partial t} &= 0, \\ \frac{\partial \Omega_j}{\partial x_i} - \frac{\partial \Omega_i}{\partial x_j} + [\Omega_i, \Omega_j] &= 0. \end{aligned}$$

It follows that

$$\begin{aligned} \left(\frac{\partial \Omega}{\partial x_i} - [\Omega, \Omega_i] \right) |_{0 \times X} &= 0, \\ d \left(\sum_i \Omega_i(0, x) dx_i \right) + \left(\sum_i \Omega_i(0, x) dx_i \right) \wedge \left(\sum_i \Omega_i(0, x) dx_i \right) &= 0. \end{aligned}$$

The second equation shows that $\sum_i \Omega_i(0, x) dx_i$ defines an integrable connection $\nabla = \nabla|_{0 \times X}$ on $\mathcal{F}|_{0 \times X}$. The first equation shows that $\Omega(0, x)$ defines a horizontal endomorphism $R_0 = \text{Res}_0(\nabla)$ of $(\mathcal{F}|_{0 \times X}, \nabla)$, which we call *the residue of ∇* . We summarize the above data as

$$\nabla = \nabla|_{0 \times X}, \quad R_0 = \text{Res}_0(\nabla) = \nabla_{t\partial_t}|_{0 \times X}, \quad \nabla \nabla = 0, \quad \nabla(R_0) = 0.$$

Poincaré rank ≤ 1 case:

The connection matrix is of the form

$$A = \frac{1}{t} \left(\sum_i \Omega_i(t, x) dx_i + \Omega(t, x) \frac{dt}{t} \right)$$

for some matrices of holomorphic functions $\Omega_i(t, x)$ and $\Omega(t, x)$. We have

$$\begin{aligned} dA + A \wedge A &= \sum_i \left(\frac{1}{t^3} [\Omega, \Omega_i] - \frac{1}{t^2} \left(\frac{\partial \Omega}{\partial x_i} + \Omega_i \right) + \frac{1}{t} \frac{\partial \Omega_i}{\partial t} \right) dt dx_i \\ &\quad \sum_{i < j} \left(\frac{1}{t^2} [\Omega_i, \Omega_j] + \frac{1}{t} \left(\frac{\partial \Omega_j}{\partial x_i} - \frac{\partial \Omega_i}{\partial x_j} \right) \right) dx_i dx_j \end{aligned}$$

From the equation $dA + A \wedge A = 0$, one deduces that

$$[\Omega, \Omega_i]|_{0 \times X} = 0, \quad [\Omega_i, \Omega_j]|_{0 \times X} = 0.$$

Let

$$\Phi = (t\nabla)|_{0 \times X} = \sum \Omega_i(0, x) dx_i : \mathcal{F}|_{0 \times X} \rightarrow \mathcal{F}|_{0 \times X} \otimes \Omega_X^1.$$

The equation $[\Omega_i, \Omega_j]|_{0 \times X} = 0$ shows that

$$\Phi \wedge \Phi = 0,$$

that is, Φ is a Higgs field on $\mathcal{F}|_{0 \times X}$. Let

$$R_0 = (t^2 \nabla_{\partial_t})|_{0 \times X} = \Omega(0, x) \in \text{End}(\mathcal{F}|_{0 \times X}).$$

The equation $[\Omega, \Omega_i]|_{0 \times X} = 0$ shows that

$$[R_0, \Phi] = 0.$$

Denote the standard coordinate on $\mathbb{A}^1 = \mathbb{P}^1 - \{\infty\}$ by t and let $s = \frac{1}{t}$. Let \mathcal{E} be a trivial vector bundle on X . Denote the sheaf of holomorphic sections of \mathcal{E} by the same notation. Let $\pi : \mathbb{P}^1 \times X \rightarrow X$ be the projection. Suppose we have an integrable connection ∇ on $\pi^*\mathcal{E}$ with a logarithmic pole along $\infty \times X$, a pole of Poincaré rank ≤ 1 along $0 \times X$, and holomorphic elsewhere. Then $\mathcal{E} \cong (\pi^*\mathcal{E})|_{\infty \times X}$ is endowed with an integrable connection

$$\nabla = \nabla|_{\infty \times X}$$

and a horizontal endomorphism

$$R_\infty = \text{Res}_\infty(\nabla),$$

and $\mathcal{E} \cong (\pi^*\mathcal{E})|_{0 \times X}$ is also endowed with a Higgs field

$$\Phi = (t\nabla)|_{0 \times X}$$

and an endomorphism

$$R_0 = (t^2\nabla_{\partial_t})|_{0 \times X}$$

commuting with Φ . The connection $(\nabla, \pi^*\mathcal{E})$ can also be constructed from the tuple $(\mathcal{E}, \nabla, R_0, R_\infty, \Phi)$. This gives rise to the so-called Frobenius type structure. More precisely, a *Frobenius type structure* (without a metric) on X is a tuple $(\mathcal{E}, \nabla, R_0, R_\infty, \Phi)$ such that \mathcal{E} is free \mathcal{O}_X -module of finite rank, $R_0, R_\infty \in \text{End}_{\mathcal{O}_X}(\mathcal{E})$, $\Phi : \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \Omega_X^1$ is a Higgs field, $\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \Omega_X^1$ is an integrable connection, and we require the following condition holds. Let $\pi : \mathbb{P}^1 \times X \rightarrow X$ be the projection. We require that the connection

$$\begin{aligned} \nabla &= \pi^*\nabla + \frac{\pi^*\Phi}{t} + \left(\frac{R_0}{t} - R_\infty\right)\frac{dt}{t} \\ &= \pi^*\nabla + s\pi^*\Phi + (-sR_0 + R_\infty)\frac{ds}{s} \end{aligned} \tag{1}$$

on $\pi^*\mathcal{E}$ is integrable. Note that ∇ has a logarithmic pole along $\infty \times X$, a pole of Poincaré rank ≤ 1 along $0 \times X$, and holomorphic elsewhere. The condition $\nabla\nabla = 0$ is equivalent to the conditions

$$\begin{aligned} \nabla\nabla &= 0, \quad \nabla(R_\infty) = 0, \quad \Phi \wedge \Phi = 0, \quad [\Phi, R_0] = 0, \\ \nabla(\Phi) &= 0, \quad \nabla(R_0) + \Phi = [\Phi, R_\infty]. \end{aligned}$$

Indeed, fix a global basis for \mathcal{E} , and let A, Φ, R_0, R_∞ be the matrix for the connection ∇ , the Higgs field Φ , the endomorphisms R_0 and R_∞ , respectively. Note that A and Φ are matrix of holomorphic 1-forms on X , and R_0 and R_∞ are matrix of holomorphic functions on X . The connection matrix for ∇ is $A' = A + \frac{\Phi}{t} + \left(\frac{R_0}{t} - R_\infty\right)\frac{dt}{t}$. The expression for $dA' + A' \wedge A'$ is

$$\begin{aligned} (dA + A \wedge A) &+ \left(\frac{1}{t^3}[\Phi, R_0] + \frac{1}{t^2}(dR_0 + [A, R_0] + \Phi - [\Phi, R_\infty]) - \frac{1}{t}(dR_\infty + [A, R_\infty])\right)dt \\ &+ \frac{1}{t^2}\Phi \wedge \Phi + \frac{1}{t}(d\Phi + A\Phi + \Phi A). \end{aligned}$$

Setting it equals to 0, we get the relations above.

Any meromorphic integrable connection ∇ on a trivial vector bundle $\pi^*\mathcal{E}$ over $\mathbb{P}^1 \times X$ with a logarithmic pole along $\infty \times X$, a pole of Poincaré ≤ 1 along $0 \times X$, and holomorphic elsewhere is

called a trTLE structure by Hertling. One can show any trTLE structure is of the form (1) and hence gives rise to a Frobenius type structure without a metric. Confer [6, Theorem 4.2]. By abuse of notation, we also call a trTLE structure a Frobenius type structure.

Birkhoff problem: Let D be a disc, and let (\mathcal{F}, ∇) be a trivial holomorphic bundle on D equipped with an integrable connection with a pole of Poincaré rank ≤ 1 at 0. Find a pair $(\tilde{\mathcal{F}}, \tilde{\nabla})$ such that $\tilde{\mathcal{F}}$ is a trivial bundle on \mathbb{P}^1 , $\tilde{\nabla}$ is an integrable meromorphic connection with logarithmic pole at ∞ and holomorphic outside $\{0, \infty\}$, and $(\tilde{\mathcal{F}}, \tilde{\nabla})|_D \cong (\mathcal{F}, \nabla)$.

Proposition 1.1 (Birkhoff problem for a family). *Let D be a disc, (X, x_0) a germ of complex manifold, and (\mathcal{F}, ∇) a trivial holomorphic bundle on $D \times X$ equipped with an integrable meromorphic connection of Poincaré rank ≤ 1 along $0 \times X$. Suppose we can solve the Birkhoff problem for $(\mathcal{F}, \nabla)|_{D \times \{x_0\}}$. Then there exists a unique pair $(\tilde{\mathcal{F}}, \tilde{\nabla})$ of a trivial holomorphic vector bundle $\tilde{\mathcal{F}}$ on $\mathbb{P}^1 \times X$ equipped with an integrable meromorphic connection $\tilde{\nabla}$ with logarithmic pole along $\infty \times X$ and holomorphic outside $\{0, \infty\} \times X$, such that $(\tilde{\mathcal{F}}, \tilde{\nabla})|_{\mathbb{P}^1 \times \{x_0\}}$ is the given solution of the Birkhoff problem, and $(\tilde{\mathcal{F}}, \tilde{\nabla})|_{D \times X} \cong (\mathcal{F}, \nabla)$. We thus get a Frobenius type structure. We have $(\tilde{\mathcal{F}}, \tilde{\nabla})|_{D_\infty \times X} \cong p^*((\tilde{\mathcal{F}}, \tilde{\nabla})|_{D_\infty \times \{x_0\}})$, where D_∞ is a disc centered at ∞ , and $p : D_\infty \times X \rightarrow D_\infty \times \{x_0\}$ is the projection.*

Algebraic version of the Birkhoff problem. Let (G, ∇) be a $\mathbb{C}[t, t^{-1}]$ -module equipped with a connection having poles only at 0, ∞ with regular singularity at ∞ , and let G_0 be a free $\mathbb{C}[t]$ -submodule of G with Poincaré rank ≤ 1 such that $G_0 \otimes_{\mathbb{C}[t]} \mathbb{C}[t, t^{-1}] = G$. Find a free $\mathbb{C}[t^{-1}]$ -submodule G_∞ of G which is logarithmic such that

$$G_0 \otimes_{\mathbb{C}[t]} \mathbb{C}[t, t^{-1}] = G_\infty \otimes_{\mathbb{C}[t^{-1}]} \mathbb{C}[t, t^{-1}] = G$$

and such that the vector bundle on \mathbb{P}^1 obtained by gluing G_0 and G_∞ is free.

Suppose the monodromy of (G, ∇) at ∞ is quasi-unipotent, and let $s = \frac{1}{t}$. Let $V_\bullet \mathbb{C}[s] \langle \partial_s \rangle$ be the increasing filtration of the Weyl algebra $\mathbb{C}[s] \langle \partial_s \rangle$ defined by

$$\begin{aligned} V_{-k} \mathbb{C}[s] \langle \partial_s \rangle &= s^k \mathbb{C}[s] \langle s \partial_s \rangle \quad \text{for } k \geq 0, \\ V_k \mathbb{C}[s] \langle \partial_s \rangle &= V_{k-1} \mathbb{C}[s] \langle \partial_s \rangle + \partial_s V_{k-1} \mathbb{C}[s] \langle \partial_s \rangle \quad \text{for } k \geq 1. \end{aligned}$$

There exists a unique increasing exhaustive filtration $V_\bullet G$ of G , indexed by a union of a finite number of subsets $\alpha + \mathbb{Z}$ ($\alpha \in \mathbb{Q}$) satisfying the following condition:

- (a) For every α , the filtration $V_{\alpha + \mathbb{Z}} G$ is good relative to $V_\bullet \mathbb{C}[s] \langle \partial_s \rangle$;
- (b) For every $\beta \in \mathbb{Q}$, $s \partial_s + \beta$ is nilpotent on $\text{Gr}_\beta^V(G) = V_\beta G / V_{<\beta} G$. Denote this nilpotent endomorphism on $\text{Gr}_\beta^V(G)$ by N .

One can show each $V_\beta G$ is a free $\mathbb{C}[s]$ -module, $\mathbb{C}[s, s^{-1}] \otimes_{\mathbb{C}[s]} V_\beta G \cong G$, and the connection has logarithmic pole on $V_\beta G$ at $s = 0$. Consider also the increasing filtration G_\bullet of G defined by $G_k = t^{-k} G_0$. This filtration induces a filtration $G_\bullet(\text{Gr}_\beta^V(G))$ on $\text{Gr}_\beta^V(G)$ for each β . Let

$$H_\infty = \bigoplus_{\beta \in [0, 1)} \text{Gr}_\beta^V(G).$$

It is the nearby cycle of (G, ∇) at ∞ .

Theorem 1.2 (M. Saito's criterion). *Suppose there exists a mixed Hodge structure on the nearby cycle H_∞ so that the Hodge filtration is $G_\bullet H_\infty$, and the weight filtration is the monodromy filtration $M_\bullet H_\infty$ of N . Then we can solve the Birkhoff problem.*

We propose the following problem:

ℓ -adic version of the Birkhoff problem. Let \mathbb{F}_q be a finite field with q elements of characteristic p , let ℓ be a prime number distinct from p , let η_0 be the generic point of the henselization of $\mathbb{P}_{\mathbb{F}_q}^1$ at 0, and let $\rho : \text{Gal}(\overline{\eta}_0/\eta_0) \rightarrow \text{GL}(n, \overline{\mathbb{Q}}_\ell)$ be a $\overline{\mathbb{Q}}_\ell$ -representation. Find conditions on ρ under which there exists a lisse punctually pure $\overline{\mathbb{Q}}_\ell$ -sheaf \mathcal{F} on $\mathbb{P}^1 - \{0, \infty\}$ tamely ramified at ∞ so that $\mathcal{F}|_{\eta_0}$ corresponds to the given representation ρ .

Let η_∞ be the generic point of the henselization of $\mathbb{P}_{\mathbb{F}_q}^1$ at ∞ . Then $\mathcal{F}_{\eta_\infty}$ considered as a representation of $\text{Gal}(\overline{\eta}_\infty/\eta_\infty)$ is the nearby cycle of \mathcal{F} at ∞ . Since \mathcal{F} is pure, the monodromy filtration on $\mathcal{F}_{\eta_\infty}$ is the weight filtration (up to a shift) by [1, 1.8.4], and this corresponds exactly to the condition of Saito's criterion. The condition that $\tilde{\mathcal{F}}$ is a trivial bundle over \mathbb{P}^1 in the classical Birkhoff problem is replaced by the condition that \mathcal{F} is a lisse punctually pure sheaf on $\mathbb{P}^1 - \{0, \infty\}$ in the ℓ -adic version.

Finally a *Frobenius type structure with a metric* on X is a tuple $(\mathcal{E}, \nabla, R_0, R_\infty, \Phi, g)$ such that $(\mathcal{E}, \nabla, R_0, R_\infty, \Phi)$ is a Frobenius type structure defined above, and $g : \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{O}_X$ is a symmetric non-degenerate ∇ -flat pairing such that the pairing

$$G : \pi^* \mathcal{E} \times a^* \pi^* \mathcal{E} \rightarrow \mathcal{O}_{\mathbb{P}^1 \times X}$$

induced by g is ∇ -flat, where $a : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ is the morphism $a(t) = -t$. This is equivalent to saying that

$$g(\Phi_v(a), b) = g(a, \Phi_v(b)), \quad g(R_0(a), b) = g(a, R_0(b)), \quad g(R_\infty(a), b) = -g(a, R_\infty(b))$$

for any tangent vector v of X and any sections a and b of \mathcal{E} . Frobenius type structures with a metric correspond to trTLEP structures of Hertling ([6, Theorem 4.2]).

2 Frobenius type structures associated to a subdiagram deformation

Let $\mathbf{w}_j = (w_{1j}, \dots, w_{nj}) \in \mathbb{Z}^n$ ($j = 1, \dots, N$), let $f = \sum_{j=1}^N a_j t_1^{w_{1j}} \cdots t_n^{w_{nj}}$ be a Laurent polynomial with nonzero coefficients $a_j \in \mathbb{C}$, and let Δ be the *Newton polyhedron* of f at ∞ , that is, the convex hull of the set $\{0, \mathbf{w}_1, \dots, \mathbf{w}_N\}$ in \mathbb{R}^n . We say f is convenient if 0 lies in the interior of Δ . We say f is *non-degenerate* if for any face σ of Δ not containing 0, the equations

$$\frac{\partial f_\sigma}{\partial t_1} = \cdots = \frac{\partial f_\sigma}{\partial t_n} = 0$$

define an empty subscheme in $\mathbb{G}_m^n = \text{Spec } \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$, where $f_\sigma = \sum_{\mathbf{w}_j \in \sigma} a_j t_1^{w_{1j}} \cdots t_n^{w_{nj}}$.

Let $g_1(t_1, \dots, t_n), \dots, g_m(t_1, \dots, t_n)$ be a family of Laurent polynomials. Consider the deformation

$$F_x(t) = f(t_1, \dots, t_n) + x_1 g_1(t_1, \dots, t_n) + \cdots + x_m g_m(t_1, \dots, t_n)$$

of f . We say F is a *subdiagram deformation* of f if all exponents of monomials with nonzero coefficients in g_1, \dots, g_m lie in the interior of Δ . We say F is the *universal unfolding* of f if the images of g_1, \dots, g_m in the Jacobian quotient ring $\mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]/\left(\frac{\partial f}{\partial t_1}, \dots, \frac{\partial f}{\partial t_n}\right)$ form a basis.

Suppose f is convenient and non-degenerate, and F is a subdiagram deformation. Consider the twisted algebraic de Rham complex

$$(\Omega_{\mathbb{G}_m^n \times \mathbb{A}^m / \mathbb{A}^m}^\bullet[\tau, \tau^{-1}], e^{\tau F} \circ d \circ e^{-\tau F}).$$

We have

$$e^{\tau F} \circ d \circ e^{-\tau F}(\omega) = d\omega - \tau dF \wedge \omega.$$

Let

$$\begin{aligned} G &= \frac{\Omega_{\mathbb{G}_m^n \times \mathbb{A}^m / \mathbb{A}^m}^n[\tau, \tau^{-1}]}{\left(d - \tau dF \wedge\right) \Omega_{\mathbb{G}_m^n \times \mathbb{A}^m / \mathbb{A}^m}^{n-1}[\tau, \tau^{-1}]}, \\ G^{(o)} &= \Omega_{\mathbb{G}_m^n}^n[\tau, \tau^{-1}] / \left(d - \tau df \wedge\right) \Omega_{\mathbb{T}^n}^{n-1}[\tau, \tau^{-1}]. \end{aligned}$$

One can show G is a free $\mathbb{C}[x_1, \dots, x_m][\tau, \tau^{-1}]$ -module of rank $n! \text{vol}(\Delta)$, and hence defines a trivial vector bundle over $(\mathbb{P}^1 - \{0, \infty\}) \times \mathbb{A}^m$. Define a connection ∇ on G and $G^{(o)}$ by

$$\nabla_{\partial_{x_j}} = e^{\tau F} \circ \partial_{x_j} \circ e^{-\tau F}, \quad \nabla_{\partial_\tau} = e^{\tau F} \circ \partial_\tau \circ e^{-\tau F}.$$

We have

$$\nabla_{\partial_{x_j}}(\omega) = \frac{\partial \omega}{\partial x_j} - \tau \frac{\partial F}{\partial x_j} \omega, \quad \nabla_{\partial_\tau}(\omega) = \frac{\partial \omega}{\partial \tau} - F\omega.$$

One can see that (G, ∇) has regular singularity at $\tau = 0$. Let $\theta = \frac{1}{\tau}$. Set

$$\begin{aligned} G_0 &= \frac{\Omega_{\mathbb{G}_m^n \times \mathbb{A}^m / \mathbb{A}^m}^n[\theta]}{\left(\theta d - dF \wedge\right) \Omega_{\mathbb{G}_m^n \times \mathbb{A}^m / \mathbb{A}^m}^{n-1}[\theta]}, \\ G_0^{(o)} &= \Omega_{\mathbb{G}_m^n}^n[\theta] / \left(\theta d - df \wedge\right) \Omega_{\mathbb{G}_m^n}^{n-1}[\theta]. \end{aligned}$$

G_0 is a free $\mathbb{C}[x_1, \dots, x_m][\theta]$ -module. It defines a trivial vector bundle over $(\mathbb{P}^1 - \{\theta = \infty\}) \times \mathbb{A}^m$. It is a lattice of G , and ∇ defines a meromorphic connection on G_0 with Poincaré rank ≤ 1 along the divisor $\theta = 0$. We call G_0 the *Brieskorn lattice* associated to the subdiagram deformation F . Using Saito's criterion, Douai and Sabbah [4] prove that the Birkhoff problem is solvable for the pairs (G, G_0) (family version) and $(G^{(o)}, G_0^{(o)})$.

Next suppose F is a universal unfolding. The definition of the pair (G, G_0) requires some analytic procedure due to the disappearance at infinity of critical points of $F_x(t)$ as $x \rightarrow 0$. Roughly speaking, G is the Fourier transform of the Gauss-Manin system for the family $F_x(t)$. There exists a neighborhood X of 0 in \mathbb{C}^m such that G is a trivial holomorphic vector bundle on $(\mathbb{P}^1 - \{0, \infty\}) \times X$ equipped with a meromorphic connection ∇ with a regular singularity along $\infty \times X$, the Brieskorn lattice G_0 is a trivial holomorphic vector bundle on $(\mathbb{P}^1 - \{\infty\}) \times X$ such that $G_0|_{(\mathbb{P}^1 - \{0, \infty\}) \times X} = G$, and the connection ∇ has Poincaré rank ≤ 1 on G_0 along $0 \times X$. When restricted to the parameter $x = 0$,

this Brieskorn lattice coincides with the one defined algebraically for the trivial deformation of f . Since we can solve the Birkhoff problem for the trivial deformation, the solution can be extended to a solution of the Birkhoff problem for the Brieskorn lattice of the universal unfolding of f . We thus get a Frobenius type structure on the universal unfolding.

To get a Frobenius manifold structure on the universal unfolding parameter space, one need to find a primitive form to transplant the Frobenius type structure to the tangent sheaf of the universal unfolding parameter space. Another approach is to start with the solution of the Birkhoff problem for a subdiagram deformation satisfying certain conditions, and then use a theorem of Hertling and Manin [6] to show that this solution has a universal deformation, which gives the Frobenius manifold structure on the universal unfolding parameter space. This is the Landau-Ginzburg B-model for the Laurent polynomial f .

In summary, we start with the Brieskorn lattice for f , which is obtained as the Fourier transform of the Gauss-Manin system associated to $f : \mathbb{G}_m^n \rightarrow \mathbb{A}^1$. Solve the Birkhoff problem for it using Saito's criterion. The Brieskorn lattice for f have a deformation, which is the Brieskorn lattice for the universal unfolding. Extend the solution of Birkhoff problem for the Brieskorn lattice associated to f to the solution of the Birkhoff problem for the Brieskorn lattice of the universal unfolding. Or we can start with the Brieskorn lattice for a subdiagram deformation. Solve the Birkhoff problem. Then apply the extension theorem of Hertling and Manin.

Arithmetically, over a finite field \mathbb{F}_q with q elements of characteristic p , we work with the Deligne-Fourier transform $\mathcal{F}(Rf_! \overline{\mathbb{Q}}_\ell)$ of $Rf_! \overline{\mathbb{Q}}_\ell$, where ℓ is a prime number distinct from p . It should satisfy arithmetic counterpart of the conditions for applying Saito's criterion, that is, it is a lisse pure sheaf on $\mathbb{P}^1 - \{0, \infty\}$. It is tamely ramified at 0, and its slopes at ∞ are ≤ 1 . Actually we can prove such kind of results for $\mathcal{F}(RF_! \overline{\mathbb{Q}}_\ell)$ for any non-degenerate deformation F of f which preserves the Newton polytope at ∞ .

3 ℓ -adic realization of the Frobenius type structures

In this section, we work over a finite ground field \mathbb{F}_q with q elements of characteristic p . Let ℓ be a prime number distinct from p . Let $\psi : \mathbb{F}_q \rightarrow \overline{\mathbb{Q}}_\ell^*$ a nontrivial additive character. For any \mathbb{F}_q -scheme S of finite type, let $D_c^b(S, \overline{\mathbb{Q}}_\ell)$ be the derived category of $\overline{\mathbb{Q}}_\ell$ -schemes on S defined in [1, 1.1.2]. For any vector bundle $E \rightarrow S$ of rank r , let E^\vee be the dual vector bundle. The *Deligne-Fourier transform* is the functor

$$\mathcal{F}_\psi : D_c^b(E, \overline{\mathbb{Q}}_\ell) \rightarrow D_c^b(E^\vee, \overline{\mathbb{Q}}_\ell), \quad K \mapsto R\mathrm{pr}_1^\vee(\mathrm{pr}^* K \otimes \langle \cdot, \cdot \rangle^* \mathcal{L}_\psi)[r],$$

where $E^\vee \rightarrow S$ is the dual vector bundle of E , $\mathrm{pr} : E \times_S E^\vee \rightarrow E$ and $\mathrm{pr}^\vee : E \times_S E^\vee \rightarrow E^\vee$ are the projections, and $\langle \cdot, \cdot \rangle : E \times_S E^\vee \rightarrow \mathbb{A}_S^1$ is the pairing, and \mathcal{L}_ψ is the Artin-Schreier sheaf associated to the nontrivial additive character ψ .

Let $\mathbf{w}_j = (w_{1j}, \dots, w_{nj}) \in \mathbb{Z}^n$ ($j = 1, \dots, N$), let $f = \sum_{j=1}^N a_j t_1^{w_{1j}} \dots t_n^{w_{nj}}$ be a Laurent polynomial with nonzero coefficients $a_j \in \mathbb{F}_q$, and let Δ be the Newton polyhedron of f at ∞ . Assume 0 lies in the interior of Δ , and assume f is non-degenerate, that is, for any face σ of Δ not containing 0, the

equations

$$\frac{\partial f_\sigma}{\partial t_1} = \dots = \frac{\partial f_\sigma}{\partial t_n} = 0$$

define an empty subscheme in $\mathbb{G}_{m, \mathbb{F}_q}^n = \text{Spec } \mathbb{F}_q[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$, where $f_\sigma = \sum_{\mathbf{w}_j \in \sigma} a_j t_1^{w_{1j}} \dots t_n^{w_{nj}}$.

Let $g_1(t_1, \dots, t_n), \dots, g_m(t_1, \dots, t_n)$ be a family of Laurent polynomials. Consider the deformation

$$F_x(t) = f(t_1, \dots, t_n) + x_1 g_1(t_1, \dots, t_n) + \dots + x_m g_m(t_1, \dots, t_n)$$

of f . Suppose the Newton polytopes of g_k ($k = 1, \dots, m$) are contained in Δ . There exists a Zariski open subset $X \subset \mathbb{A}^m$ containing the origin so that for any $\overline{\mathbb{F}}_q$ -point (x_1, \dots, x_m) in X , the Newton polyhedron of $F_x(t)$ at ∞ coincides with Δ , and $F_x(t)$ is non-degenerate with respect to Δ . Consider the morphism

$$F : \mathbb{G}_{m, X}^n \rightarrow \mathbb{A}_X^1, \quad (t, x) \mapsto (F_x(t), x).$$

The Deligne-Fourier transform $\mathcal{F}_\psi(RF_! \overline{\mathbb{Q}}_\ell)$ of $RF_! \overline{\mathbb{Q}}_\ell$ is an analogue of G in §2. Here the Deligne-Fourier transform is taken for the vector bundle $\mathbb{A}_X^1 \rightarrow X$. Note that we have

$$\mathcal{F}_{\psi^{-1}}(RF_! \overline{\mathbb{Q}}_\ell) \cong a^* \mathcal{F}_\psi(RF_! \overline{\mathbb{Q}}_\ell),$$

where $a : \mathbb{A}_X^1 \rightarrow \mathbb{A}_X^1$ is the morphism $\tau \mapsto -\tau$.

Theorem 3.1. *Notation as above. Suppose the Newton polytopes of g_i are contained in Δ .*

(i) *When restricted to $(\mathbb{P}^1 - \{0, \infty\}) \times X$, $\mathcal{H}^i(\mathcal{F}_\psi(RF_! \overline{\mathbb{Q}}_\ell)) = 0$ for $i \neq n-1$, and $\mathcal{H}^{n-1}(\mathcal{F}_\psi(RF_! \overline{\mathbb{Q}}_\ell))$ is a pure lisse sheaf of weight n .*

(ii) *When restricted to $\mathbb{P}_X^1 - (\{0, \infty\} \times X)$, we have a perfect pairing*

$$\mathcal{H}^{n-1}(\mathcal{F}_\psi(RF_! \overline{\mathbb{Q}}_\ell)) \times \mathcal{H}^{n-1}(\mathcal{F}_{\psi^{-1}}(RF_! \overline{\mathbb{Q}}_\ell)) \rightarrow \overline{\mathbb{Q}}_\ell(-n).$$

(iii) *For each fixed geometric point x of X , the restriction of $\mathcal{H}^{n-1}(\mathcal{F}_\psi(RF_! \overline{\mathbb{Q}}_\ell))$ to $(\mathbb{P}^1 - \{0, \infty\}) \times \{x\}$ is tamely ramified at 0, and has slopes ≤ 1 at ∞ .*

Let K be a local field with perfect residue field. We have an equivalence between the category of $\overline{\mathbb{Q}}_\ell$ -sheaves on $\text{Spec } K$ and the category of $\overline{\mathbb{Q}}_\ell$ -representations of $\text{Gal}(\overline{K}/K)$. The higher ramification subgroups of $\text{Gal}(\overline{K}/K)$ give rise to a decreasing filtration in upper numbering $\{G^{(r)}\}_{r \in \mathbb{Q}_{\geq 0}}$. Confer [9]. Let $G^{(r+)}$ be the closure of $\cup_{s > r} G^{(s)}$. Then $G^{(0+)}$ is the wild ramification subgroup of $\text{Gal}(\overline{K}/K)$. Any $\overline{\mathbb{Q}}_\ell$ -representation V of $\text{Gal}(\overline{K}/K)$ is semisimple as a representation of $G^{(0+)}$. Let $V = \oplus_i V_i$ be a decomposition of V into irreducible representations of $G^{(0+)}$. The slope of V_i is the smallest rational number s_i such that $V_i^{G^{(s_i+)}} = V_i$. The numbers s_i are called slopes of V .

Let \mathbb{F} be the algebraic closure of \mathbb{F}_q , and let $\mathbb{F}[[\pi]]$ be the formal power series ring. The field of fractions of $\mathbb{F}[[\pi]]$ is the field $\mathbb{F}((\pi))$ of formal Laurent series. On $\text{Spec } \mathbb{F}[[\pi]]$, let o be the divisor defined by the maximal ideal of $\mathbb{F}[[\pi]]$. Let \mathcal{H} be a lisse sheaf on $(\mathbb{P}^1 - \{0, \infty\}) \times X$. We say \mathcal{H} is *tamely ramified* at $0 \times X$ if for any \mathbb{F}_q -morphism $g : \text{Spec } \mathbb{F}[[\pi]] \rightarrow \mathbb{P}^1 \times X$ such that $g^*(0 \times X)$ is the divisor eo ($e \geq 1$), the sheaf $(g^* \mathcal{H})|_{\text{Spec } \mathbb{F}((\pi))}$ is tamely ramified. We say \mathcal{H} has *slope* $\leq r$ at $\infty \times X$ if for any \mathbb{F}_q -morphism $g : \text{Spec } \mathbb{F}[[\pi]] \rightarrow \mathbb{P}^1 \times X$ such that $g^*(\infty \times X)$ is the divisor eo ($e \geq 1$), the

sheaf $(g^*\mathcal{H})|_{\mathrm{Spec}\mathbb{F}((\pi))}$ has slopes $\leq er$. We can also define slopes using Abbes-Saito's theory for higher ramifications of Galois representations of local field with imperfect residue field.

In view of the fact that a Frobenius type structure has logarithmic pole at $\infty \times X$, and has Poincaré rank ≤ 1 at $0 \times X$, the following fact which is more general than Theorem 3.1(iii) should be true: $\mathcal{H}^{n-1}(\mathcal{F}_\psi(RF_!\overline{\mathbb{Q}}_\ell))$ is tamely ramified at $0 \times X$, and has slopes ≤ 1 at $\infty \times X$.

We have seen that connections of Poincaré rank ≤ 1 gives rise to structures such as Higgs fields. Due to our lack of explicit description of the higher ramifications, we haven't been able to extract structures hidden in ℓ -adic sheaves of slope ≤ 1 along a divisor.

We define an ℓ -adic Frobenius type structure with a metric to be a pure lisse sheaf \mathcal{H} on $(\mathbb{P}^1 - \{0, \infty\}) \times X$ which is tamely ramified at $\infty \times X$, and has slopes ≤ 1 at $0 \times X$, and has a perfect pairing

$$\mathcal{H} \times a^*\mathcal{H} \rightarrow \overline{\mathbb{Q}}_\ell(-n),$$

where n is the weight of \mathcal{H} . By the above discussion, $\mathrm{inv}^*(\mathcal{H}^{n-1}(\mathcal{F}_\psi(RF_!\overline{\mathbb{Q}}_\ell)))$ should define an ℓ -adic Frobenius type structure, where $\mathrm{inv} : (\mathbb{P}^1 - \{0, \infty\}) \times X \rightarrow (\mathbb{P}^1 - \{0, \infty\}) \times X$ is the morphism defined by $\tau \mapsto \frac{1}{\tau}$.

The proof of Theorem 3.1 uses the properties of ℓ -adic Gelfand-Kapranov-Zelevinsky (GKZ) hypergeometric sheaves introduced in [5]. Choose $\mathbf{w}_{N+1}, \dots, \mathbf{w}_{N'}$ so that

$$\Delta \cap \mathbb{Z}^n = \{\mathbf{w}_1, \dots, \mathbf{w}_N, \mathbf{w}_{N+1}, \dots, \mathbf{w}_{N'}\}.$$

Let $\pi_2 : \mathbb{G}_m^n \times \mathbb{A}^{N'} \rightarrow \mathbb{A}^{N'}$ be the projection, and let H be the morphism

$$H : \mathbb{G}_m^n \times \mathbb{A}^{N'} \rightarrow \mathbb{A}^1, \quad (t_1, \dots, t_n, y_1, \dots, y_{N'}) \mapsto \sum_{j=1}^{N'} y_j t_1^{w_{1j}} \dots t_n^{w_{nj}}.$$

We define the ℓ -adic GKZ hypergeometric sheaf to be the object in $D_c^b(\mathbb{A}^{N'}, \overline{\mathbb{Q}}_\ell)$ defined by

$$\mathrm{Hyp}_\psi = R\pi_{2!}H^*\mathcal{L}_\psi[n + N'].$$

We have the following:

Theorem 3.2.

- (i) Hyp_ψ is a pure perverse sheaf on $\mathbb{A}^{N'}$ of weight $n + N'$ and of rank $(-1)^{N'} n! \mathrm{vol}(\Delta)$.
- (ii) Suppose V is a Zariski open subset of $\mathbb{A}^{N'}$ such that for any $(a_1, \dots, a_{N'}) \in V(\overline{\mathbb{F}}_q)$, the Laurent polynomial $\sum_{j=1}^{N'} a_j t_1^{w_{1j}} \dots t_n^{w_{nj}}$ is nondegenerate with respect to Δ . Then $\mathcal{H}^i(\mathrm{Hyp})|_V = 0$ for $i \neq -N'$, and $\mathcal{H}^{-N'}(\mathrm{Hyp}_\psi)|_V$ is lisse, pure of weight n , and of rank $n! \mathrm{vol}(\Delta)$.
- (iii) The Verdier dual $D(\mathrm{Hyp}_\psi) := R\mathcal{H}om(\mathrm{Hyp}_\psi, \overline{\mathbb{Q}}_\ell(N')[2N'])$ of Hyp_ψ is isomorphic to $\mathrm{Hyp}_{\psi^{-1}}(n + N')$, where $(n + N')$ denotes the Tate twist by $\overline{\mathbb{Q}}_\ell(n + N')$. In particular, on the open set V , we have a perfect pairing

$$\mathcal{H}^{-N'}(\mathrm{Hyp}_\psi) \times \mathcal{H}^{-N'}(\mathrm{Hyp}_{\psi^{-1}}) \rightarrow \overline{\mathbb{Q}}_\ell(-n).$$

The main technique to study the ℓ -adic GKZ hypergeometric sheaf is again the Deligne-Fourier transform. Let $\iota : \mathbb{T}^n \rightarrow \mathbb{A}^{N'}$ be the morphism defined by

$$(t_1, \dots, t_n) \mapsto (t_1^{w_{1j}} \dots t_n^{w_{nj}})_{j=1, \dots, N'}.$$

In [5], we prove that

$$\mathrm{Hyp}_\psi = \mathcal{F}_\psi(\iota_! \overline{\mathbb{Q}}_\ell[n]),$$

where \mathcal{F}_ψ is the Deligne-Fourier transform for the vector bundle $\mathbb{A}^{N'} \rightarrow \mathrm{Spec} \mathbb{F}_q$. Using the assumption that 0 lies in the interior Δ , one can show $\iota_! \overline{\mathbb{Q}}_\ell[n] = \iota_{!*} \overline{\mathbb{Q}}_\ell[n]$. From the standard facts on perverse sheaves and the Deligne-Fourier transform, one deduces that Hyp_ψ is a pure perverse sheaf of weight $n + N'$, and $D(\mathrm{Hyp}_\psi) \cong \mathrm{Hyp}_{\psi^{-1}}(n + N')$. The other statements in Theorem 3.2 require a detailed study of the morphism H relatively to the toric compactification defined by the convex polytope Δ .

Write

$$F_x(t) = f(t_1, \dots, t_n) + x_1 g_1(t_1, \dots, t_n) + \dots + x_m g_m(t_1, \dots, t_n) = \sum_{j=1}^{N'} y_j(x_1, \dots, x_m) t_1^{w_{1j}} \dots t_n^{w_{nj}}.$$

Then $y_j(x_1, \dots, x_m)$ are linear polynomial of x_1, \dots, x_m with coefficients depending on f and g_1, \dots, g_m . Consider the morphism

$$\Phi : \mathbb{A}_X^1 \rightarrow \mathbb{A}^{N'}, \quad (\tau, x_1, \dots, x_m) \rightarrow (\tau y_1(x_1, \dots, x_m), \dots, \tau y_{N'}(x_1, \dots, x_m)).$$

Theorem 3.1 follows from Theorem 3.2 and the following:

Proposition 3.3.

- (i) We have $\mathcal{F}_\psi(RF_! \overline{\mathbb{Q}}_\ell) \cong \Phi^*(\mathrm{Hyp}_\psi)[1 - (n + N')]$.
- (ii) The image of $(\mathbb{P}^1 - \{0, \infty\}) \times X$ under Φ is contained in the set V parameterizing non-degenerate Laurent polynomials.

Instead of working with Laurent polynomials, one can also work with polynomials, and similar results still hold.

Originally our motivation for studying the ℓ -adic GKZ hypergeometric sheaf comes from the study of exponential sums using ℓ -adic cohomology theory. For any \mathbb{F}_q -rations points $x = (x_1, \dots, x_{N'})$ of $\mathbb{A}^{N'}$, it follows from the Grothendieck trace formula that we have

$$(-1)^{n+N} \mathrm{Tr}(\mathrm{Fr}_x, (\mathrm{Hyp}_\psi)_{\bar{x}}) = \sum_{t_1, \dots, t_n \in \mathbb{F}_q^*} \psi\left(\sum_{j=1}^{N'} y_j t_1^{w_{1j}} \dots t_n^{w_{nj}}\right), \quad (2)$$

where the left hand side is the trace of the geometric Frobenius at the point x acting on the stalk of Hyp_ψ at the geometric point \bar{x} above x . Note that the right hand side of the above equation is a family of exponential sums parameterized by $(y_1, \dots, y_{N'})$. It is an analogue of the oscillatory integral

$$\int_\sigma e^{i \sum_{j=1}^{N'} y_j t_1^{w_{1j}} \dots t_n^{w_{nj}}} \frac{dt_1}{t_1} \dots \frac{dt_n}{t_n},$$

where σ is an n -dimensional cycle in $\mathbb{G}_{m, \mathbb{C}}^n$. This integral is a solution of the GKZ hypergeometric system of differential equations.

Similarly, for any \mathbb{F}_q -rational point $(\tau, x) = (\tau, x_1, \dots, x_m)$ of \mathbb{A}_X^1 , we have

$$\begin{aligned} & -\mathrm{Tr}\left(\mathrm{Fr}_{(\tau, x)}, (\mathcal{F}_\psi(RF_! \overline{\mathbb{Q}}_\ell))_{(\tau, x)}\right) \\ &= \sum_{t_1, \dots, t_n \in \mathbb{F}_q^*} \psi\left(\tau(f(t_1, \dots, t_n) + x_1 g_1(t_1, \dots, t_n) + \dots + x_m g_m(t_1, \dots, t_n))\right). \end{aligned} \quad (3)$$

It is clear that the family of exponential sum on the right hand side of (3) is the composite of Φ and the family of exponential sum on the right hand side of (2). This gives an explanation of Proposition 3.3 (i) on the level of functions. The oscillatory integral corresponding to the exponential sum in (3) is

$$\int_{\sigma} e^{i\tau(f(t_1, \dots, t_n) + x_1 g_1(t_1, \dots, t_n) + \dots + x_m g_m(t_1, \dots, t_n))} \frac{dt_1}{t_1} \dots \frac{dt_n}{t_n}.$$

It is a solution of the system of differential equations of the D -module defined by the connection introduced in §2.

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